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On the Location of the Zeros of a Polynomial

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The classical Eneström–Kekeya Theorem states that a polynomial $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ satisfying $0 < a_0 \le a_1 \le \cdots \le a_n$ has all its zeros in $|z| \le 1$. We extend this result to a larger class of polynomials by dropping the conditions that the coefficients be real and positive and weakening the hypothesis that coefficients be monotonic increasing. Our result generalizes and sharpens several known results. © 1994 Academic Press, Inc.

1. INTRODUCTION AND STATEMENT OF RESULTS

Many results exist concerning the location of the zeros of a polynomial of a complex variable as a function of the coefficients of the polynomial. One such classical result is the following well known Eneström–Kakeya Theorem.

THEOREM A [2, 4]. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n with real coefficients satisfying*

$$0 < a_0 \le a_1 \le \cdots \le a_n,$$

then all the zeros of $p(z)$ lie in $|z| \le 1$.

Joyal *et al.* [3] dropped the hypothesis that the coefficients be all positive and proved the following generalization of Theorem A.

THEOREM B. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n with real coefficients, $a_n \ne 0$, satisfying*

$$a_0 \le a_1 \le a_2 \le \cdots \le a_n,$$

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then all the zeros of $p(z)$ *lie in*

$$|z| \leq \frac{a_n - a_0 + |a_0|}{a_n}.$$

We wish to weaken the hypothesis of Theorem B and consider a larger class of polynomials. We are inspired by the following theorem of Aziz and Mohammad [1], which is for analytic functions.

THEOREM C. *Let* $f(z) = \sum_{i=0}^{\infty} a_i z^i$ *be analytic in* $|z| \leq t$. *If* $\operatorname{Re} a_j = \alpha_j$ *and* $\operatorname{Im} a_j = \beta_j$ *for* $j = 0, 1, 2, \ldots$ *and for some* k *and* r,

$$0 < \alpha_0 \leq t\alpha_1 \leq \cdots \leq t^k \alpha_k \geq t^{k+1} \alpha_{k+1} \geq \ldots,$$

and

$$\beta_0 \leq t\beta_1 \leq \ldots t^r \beta_r \geq t^{r+1} \beta_{r+1} \geq \ldots,$$

then $f(z)$ *has all its zeros in*

$$|z| \geq \frac{t|a_0|}{2(\alpha_k t^k + \beta_r t^r) - (\alpha_0 + \beta_0)}.$$

For polynomials we can prove the following more general result. The interest of this theorem also lies in its flexibility and this is demonstrated in the four corollaries that follow from it.

THEOREM. *Let* $p(z) = \sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* n. *If* $\operatorname{Re} a_j = \alpha_j$ *and* $\operatorname{Im} a_j = \beta_j$ *for* $j = 0, 1, 2, \ldots, n$, $a_n \neq 0$ *and for some* k *and* r *and for some* $t \geq 0$,

$$\alpha_0 \leq t\alpha_1 \leq t^2\alpha_2 \leq \cdots \leq t^k \alpha_k \geq t^{k+1} \alpha_{k+1} \geq \cdots \geq t^n \alpha_n,$$

and

$$\beta_0 \leq t\beta_1 \leq t^2\beta_2 \leq \cdots \leq t^r \beta_r \geq t^{r+1} \beta_{r+1} \geq \cdots \geq t^n \beta_n,$$

then $p(z)$ *has all its zeros in* $R_1 \leq |z| \leq R_2$, *where*

$$R_1 = \min\{(t|a_0| / (2(t^k\alpha_k + t^r\beta_r) - (\alpha_0 + \beta_0) - t^n(\alpha_n + \beta_n - |a_n|)), t\}$$

and

$$R_2 = \max\left\{\left(|a_0|t^{n+1} - t^{n-1}(\alpha_0 + \beta_0) - t(\alpha_n + \beta_n)\right.\right.$$

$$+(t^2 + 1)\left(t^{n-k-1}\alpha_k + t^{n-r-1}\beta_r\right)$$

$$+(t^2 - 1)\left(\sum_{j=1}^{k-1} t^{n-j-1}\alpha_j + \sum_{j=1}^{r-1} t^{n-j-1}\beta_j\right)$$

$$\left.\left. +(1 - t^2)\left(\sum_{j=k+1}^{n-1} t^{n-j-1}\alpha_j + \sum_{j=r+1}^{n-1} t^{n-j-1}\beta_j\right)\right/|a_n|, \frac{1}{t}\right\},$$

Remark. Note that the above theorem is more general than Theorem C when applied to polynomials. For polynomials, Theorem C requires $\alpha_j \geq 0$ for all j, whereas our theorem does not require this hypothesis. Also when applied to an admissible polynomial, the bound obtained by our theorem may be considerably better than the bound obtained from Theorem C. We illustrate this by the following example.

EXAMPLE.

$$p(z) = (1 + i) + (0.2 + 0.2i)z + (0.03 + 0.03i)z^2$$
$$+(0.0031 + 0.0031i)z^3 + (0.0003 + 0.0003i)z^4.$$

Theorem C gives that $p(z)$ has all its zeros in $|z| \geq 1.3598$, while by our theorem when applied to $p(z)$ with $t = 10$, $k = r = 3$, we find that $p(z)$ has all is zeros in $|z| \geq 1.6363$, an improvement on the bound of Theorem C by over 20%. Besides if we also care to calculate the outer radius R_2, we will get an annulus containing all the zeros of $p(z)$.

If in our theorem, we take $t = 1$, $k = n$ and $r = n$, we get:

COROLLARY 1. *Let* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$, $a_n \neq 0$, *and* $\operatorname{Re} a_j = \alpha_j$ *and* $\operatorname{Im} a_j = \beta_j$ *for* $j = 0, 1, 2, \ldots, n$. *If*

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \quad \text{and} \quad \beta_0 \leq \beta_1 \leq \cdots \leq \beta_n,$$

then $p(z)$ *has all its zeros in*

$$\frac{|a_0|}{|a_n| - (\alpha_0 + \beta_0) + (\alpha_n + \beta_n)} \leq |z| \leq \frac{|a_0| - (\alpha_0 + \beta_0) + (\alpha_n + \beta_n)}{|a_n|}.$$

In particular, if all the coefficients a_j are real, that is, $\beta_j = 0$ for $0 \le j \le n$, Corollary 1 gives that $p(z)$ has all its zeros in the annulus

$$\frac{|a_0|}{|a_n| - a_0 + a_n} \le |z| \le \frac{|a_0| - a_0 + a_n}{|a_n|},$$

which is an improvement of Theorem B. Further if the coefficients a_j are all non-negative, $a_n \ne 0$ then Corollary 1 clearly reduces to Theorem A, the Eneström–Kakeya Theorem.

By making suitable choices of t, k, and r, one can, in fact, obtain the following corollaries which are also of interest. In each of these $p(z) = \sum_{\iota=0}^{n} a_\iota z^\iota$, $a_n \ne 0$, and Re $a_j = \alpha_j$ and Im $a_j = \beta_j$ for $j = 0, 1, 2, \ldots, n$.

COROLLARY 2. *If $\alpha_0 \ge \alpha_1 \ge \cdots \ge \alpha_n$ and $\beta_0 \ge \beta_1 \ge \cdots \ge \beta_n$, then $p(z)$ has all its zeros in*

$$\frac{|a_0|}{|a_n| + (\alpha_0 + \beta_0) - (\alpha_n + \beta_n)} \le |z| \le \frac{|a_0| + (\alpha_0 + \beta_0) - (\alpha_n + \beta_n)}{|a_n|}.$$

COROLLARY 3. *If $\alpha_0 \ge \alpha_1 \ge \cdots \ge \alpha_n$ and $\beta_0 \le \beta_1 \le \cdots \le \beta_n$, then $p(z)$ has all its zeros in*

$$\frac{|a_0|}{|a_n| + \alpha_0 - \beta_0 - \alpha_n + \beta_n} \le |z| \le \frac{|a_0| + \alpha_0 - \beta_0 - \alpha_n + \beta_n}{|a_n|}.$$

COROLLARY 4. *If $\alpha_0 \le \alpha_1 \le \cdots \le \alpha_n$ and $\beta_0 \ge \beta_1 \ge \cdots \ge \ldots \beta_n$, then $p(z)$ has all its zeros in*

$$\frac{|a_0|}{|a_n| - \alpha_0 + \beta_0 + \alpha_n - \beta_n} \le |z| \le \frac{|a_0| - \alpha_0 + \beta_0 + \alpha_n - \beta_n}{|a_n|}.$$

Note that one can obtain Corollaries 2, 3, and 4 from Corollary 1 as well.

2. PROOF OF THE THEOREM

Consider the polynomial

$$\begin{aligned} P(z) &= (t - z)p(z) \\ &= ta_0 + \sum_{j=1}^{n} (ta_j - a_{j-1}) z^j - a_n z^{n+1} \\ &= ta_0 + G_1(z), \qquad \text{say.} \end{aligned}$$

On $|z| = t$,

$$
\begin{aligned}
|G_1(z)| &\le \sum_{j=1}^{n} |ta_j - a_{j-1}| t^j + |a_n| t^{n+1} \\
&\le \sum_{j=1}^{k} (t\alpha_j - \alpha_{j-1}) t^j + \sum_{j=k+1}^{n} (\alpha_{j-1} - t\alpha_j) t^j \\
&\quad + \sum_{j=1}^{r} (t\beta_j - \beta_{j-1}) t^j + \sum_{j=r+1}^{n} (\beta_{j-1} - t\beta_j) t^j + |a_n| t^{n+1} \\
&= -t(\alpha_0 + \beta_0) + 2\left(t^{k+1}\alpha_k + t^{r+1}\beta_r\right) - t^{n+1}(\alpha_n + \beta_n - |a_n|) \\
&= M_1,
\end{aligned}
$$

and on applying Schwarz's lemma [5, p. 168] to $G_1(z)$, we get

$$
|G_1(z)| \le \frac{M_1 |z|}{t}, \qquad \text{for } |z| \le t,
$$

which implies

$$
\begin{aligned}
|P(z)| &= |-ta_0 + G_1(z)| \\
&\ge t|a_0| - |G_1(z)| \\
&\ge t|a_0| - \frac{M_1|z|}{t}, \qquad \text{for } |z| \le t.
\end{aligned}
$$

Hence, if $|z| < R_1 = \min\{(t^2|a_0|/M_1), t\}$, then $P(z) \neq 0$ and so $p(z) \neq 0$.

Next we show that $p(z) \neq 0$ if $|z| > R_2$. For this, we again consider

$$
\begin{aligned}
P(z) = (t - z)p(z) &= ta_0 + \sum_{j=1}^{n} (ta_j - a_{j-1}) z^j - a_n z^{n+1} \\
&= -a_n z^{n+1} + G_2(z), \qquad \text{say}.
\end{aligned}
$$

Then

$$
\left| z^n G_2\left(\frac{1}{z}\right) \right| = \left| ta_0 z^n + \sum_{j=1}^{n} (ta_j - a_{j-1}) z^{n-j} \right|,
$$

and on $|z| = t$,

$$\begin{aligned}\left|z^n G_2\left(\frac{1}{z}\right)\right| &\le t|a_0|t^n + \sum_{j=1}^{n} |ta_j - a_{j-1}|t^{n-j} \\ &\le |a_0|t^{n+1} + \sum_{j=1}^{n} |t\alpha_j - \alpha_{j-1}|t^{n-j} + \sum_{j=1}^{n} \left(|t\beta_j - t\beta_{j-1}|\right)t^{n-j} \\ &= |a_0|t^{n+1} - t^{n-1}(\alpha_0 + \beta_0) - t(\alpha_n + \beta_n) \\ &\quad + (t^2 + 1)\left(t^{n-k-1}\alpha_k + t^{n-r-1}\beta_r\right) \\ &\quad + (t^2 - 1)\left(\sum_{j=1}^{k-1} t^{n-j-1}\alpha_j + \sum_{j=1}^{r-1} t^{n-j-1}\beta_j\right) \\ &\quad + (1 - t^2)\left(\sum_{j=k+1}^{n-1} t^{n-j-1}\alpha_j + \sum_{j=r+1}^{n-1} t^{n-j-1}\beta_j\right) \\ &= M_2.\end{aligned}$$

Hence it follows by the maximum modulus theorem [5, p. 165] that

$$\left|z^n G_2\left(\frac{1}{z}\right)\right| \le M_2, \qquad \text{for } |z| \le t,$$

which implies

$$|G_2(z)| \le M_2|z|^n, \qquad \text{for } |z| \ge \frac{1}{t}.$$

From this it follows that

$$\begin{aligned}|P(z)| &= |-a_n z^{n+1} + G_2(z)| \\ &\ge |a_n|\,|z|^{n+1} - M_2|z|^n, \qquad \text{for } |z| \ge 1/t, \\ &= |z|^n(|a_n|\,|z| - M_2).\end{aligned}$$

Thus if $|z| > R_2 = \max\{(M_2/|a_n|), (1/t)\}$, then $P(z) \neq 0$ and hence $p(z) \neq 0$, and the proof of the theorem is complete.

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